

NOTE ON CERTAIN INEQUALITIES FOR NEUMAN MEANS

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ABSTRACT. In this paper, we give the explicit formulas for the Neuman means N_{AH} , N_{HA} , N_{AC} and N_{CA} , and present the best possible upper and lower bounds for these means in terms of the combinations of harmonic mean H , arithmetic mean A and contraharmonic mean C .

1. INTRODUCTION

Let $a, b, c \geq 0$ with $ab + ac + bc \neq 0$. Then the symmetric integral $R_F(a, b, c)$ [1] of the first kind is defined as

$$R_F(a, b, c) = \frac{1}{2} \int_0^\infty [(t+a)(t+b)(t+c)]^{-1/2} dt.$$

The degenerate case of R_F , denoted by R_C plays an important role in the theory of special functions [1, 2], which is given by

$$R_C(a, b) = R_F(a, b, b).$$

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a, b)$ [3-5] of a and b is given by

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \frac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

Carson [6] (see also [7, (3.21)]) proved that

$$SB(a, b) = [R_C(a^2, b^2)]^{-1}.$$

Recently, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for the Schwab-Borchardt mean and it generated means can be found in the literature [3-5, 8-11].

Let $a > b > 0$, $v = (a - b)/(a + b) \in (0, 1)$, $p \in (0, \infty)$, $q \in (0, \pi/2)$, $r \in (0, \log(2 + \sqrt{3}))$ and $s \in (0, \pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$, $H(a, b) = 2ab/(a + b)$, $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$, $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ and $C(a, b) = (a^2 + b^2)/(a + b)$ be respectively the harmonic, geometric, arithmetic, quadratic and contraharmonic means of a and b , $S_{AH}(a, b) = SB[A(a, b), H(a, b)]$, $S_{HA}(a, b) = SB[H(a, b), A(a, b)]$, $S_{AC}(a, b) = SB[A(a, b), C(a, b)]$, $S_{CA}(a, b) = SB[C(a, b), A(a, b)]$. Then Neuman [10] gave the explicit formulas

$$(1.1) \quad S_{AH}(a, b) = A(a, b) \frac{\tanh(p)}{p}, \quad S_{HA}(a, b) = A(a, b) \frac{\sin q}{q},$$

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$$(1.2) \quad S_{CA}(a, b) = A(a, b) \frac{\sinh(r)}{r}, \quad S_{AC}(a, b) = A(a, b) \frac{\tan s}{s}.$$

Very recently, Neuman [12] found a new mean $N(a, b)$ derived from the Schwab-Borchardt mean as follows:

$$(1.3) \quad N(a, b) = \frac{1}{2} \left[a + \frac{b^2}{SB(a, b)} \right].$$

Let $N_{AH}(a, b) = N[A(a, b), H(a, b)]$, $N_{HA}(a, b) = N[H(a, b), A(a, b)]$, $N_{AG}(a, b) = N[A(a, b), G(a, b)]$, $N_{GA}(a, b) = N[G(a, b), A(a, b)]$, $N_{AC}(a, b) = N[A(a, b), C(a, b)]$, $N_{CA}(a, b) = N[C(a, b), A(a, b)]$, $N_{AQ}(a, b) = N[A(a, b), Q(a, b)]$ and $N_{QA}(a, b) = N[Q(a, b), A(a, b)]$ be the Neuman means. Then Neuman [12] proved that

$$G(a, b) < N_{AG}(a, b) < N_{GA}(a, b) < A(a, b) < N_{QA}(a, b) < N_{AQ}(a, b) < Q(a, b)$$

for all $a, b > 0$ with $a \neq b$, and the double inequalities

$$\begin{aligned} \alpha_1 A(a, b) + (1 - \alpha_1) G(a, b) &< N_{GA}(a, b) < \beta_1 A(a, b) + (1 - \beta_1) G(a, b), \\ \alpha_2 Q(a, b) + (1 - \alpha_2) A(a, b) &< N_{AQ}(a, b) < \beta_2 Q(a, b) + (1 - \beta_2) A(a, b), \\ \alpha_3 A(a, b) + (1 - \alpha_3) G(a, b) &< N_{AG}(a, b) < \beta_3 A(a, b) + (1 - \beta_3) G(a, b), \\ \alpha_4 Q(a, b) + (1 - \alpha_4) A(a, b) &< N_{QA}(a, b) < \beta_4 Q(a, b) + (1 - \beta_4) A(a, b) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 2/3$, $\beta_2 \geq (\pi - 2)/[4(\sqrt{2} - 1)] = 0.689\dots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq 1/2$, $\alpha_4 \leq 1/3$ and $\beta_4 \geq [\log(1 + \sqrt{2}) + \sqrt{2} - 2]/[2(\sqrt{2} - 1)] = 0.356\dots$

Zhang et. al. [13] presented the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0, 1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2, 1]$ such that the double inequalities

$$\begin{aligned} G(\alpha_1 a + (1 - \alpha_1)b, \alpha_1 b + (1 - \alpha_1)a) &< N_{AG}(a, b) < G(\beta_1 a + (1 - \beta_1)b, \beta_1 b + (1 - \beta_1)a), \\ G(\alpha_2 a + (1 - \alpha_2)b, \alpha_2 b + (1 - \alpha_2)a) &< N_{GA}(a, b) < G(\beta_2 a + (1 - \beta_2)b, \beta_2 b + (1 - \beta_2)a), \\ Q(\alpha_3 a + (1 - \alpha_3)b, \alpha_3 b + (1 - \alpha_3)a) &< N_{QA}(a, b) < Q(\beta_3 a + (1 - \beta_3)b, \beta_3 b + (1 - \beta_3)a), \\ Q(\alpha_4 a + (1 - \alpha_4)b, \alpha_4 b + (1 - \alpha_4)a) &< N_{AQ}(a, b) < Q(\beta_4 a + (1 - \beta_4)b, \beta_4 b + (1 - \beta_4)a) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

In [14], the authors found the greatest values $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ and the least values $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$ such that the double inequalities

$$\begin{aligned} A^{\alpha_1}(a, b) G^{1-\alpha_1}(a, b) &< N_{GA}(a, b) < A^{\beta_1}(a, b) G^{1-\beta_1}(a, b), \\ \frac{\alpha_2}{G(a, b)} + \frac{1 - \alpha_2}{A(a, b)} &< \frac{1}{N_{GA}(a, b)} < \frac{\beta_2}{G(a, b)} + \frac{1 - \beta_2}{A(a, b)}, \\ A^{\alpha_3}(a, b) G^{1-\alpha_3}(a, b) &< N_{AG}(a, b) < A^{\beta_3}(a, b) G^{1-\beta_3}(a, b), \\ \frac{\alpha_4}{G(a, b)} + \frac{1 - \alpha_4}{A(a, b)} &< \frac{1}{N_{AG}(a, b)} < \frac{\beta_4}{G(a, b)} + \frac{1 - \beta_4}{A(a, b)}, \\ Q^{\alpha_5}(a, b) A^{1-\alpha_5}(a, b) &< N_{AQ}(a, b) < Q^{\beta_5}(a, b) A^{1-\beta_5}(a, b), \\ \frac{\alpha_6}{A(a, b)} + \frac{1 - \alpha_6}{Q(a, b)} &< \frac{1}{N_{AQ}(a, b)} < \frac{\beta_6}{A(a, b)} + \frac{1 - \beta_6}{Q(a, b)}, \\ Q^{\alpha_7}(a, b) A^{1-\alpha_7}(a, b) &< N_{QA}(a, b) < Q^{\beta_7}(a, b) A^{1-\beta_7}(a, b), \\ \frac{\alpha_8}{A(a, b)} + \frac{1 - \alpha_8}{Q(a, b)} &< \frac{1}{N_{QA}(a, b)} < \frac{\beta_8}{A(a, b)} + \frac{1 - \beta_8}{Q(a, b)} \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to give the explicit formulas for the Neuman means N_{AH} , N_{HA} , N_{AC} and N_{CA} , and present the best possible upper and lower

bounds for theses means in terms of the combinations of harmonic, arithmetic and contraharmonic means. Our main results are the following Theorems 1.1-1.3.

Theorem 1.1. *Let $a > b > 0$, $v = (a - b)/(a + b) \in (0, 1)$, $p \in (0, \infty)$, $q \in (0, \pi/2)$, $r \in (0, \log(2 + \sqrt{3}))$ and $s \in (0, \pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$. Then we have*

$$(1.4) \quad N_{AH}(a, b) = \frac{1}{2}A(a, b) \left[1 + \frac{2p}{\sinh(2p)} \right],$$

$$(1.5) \quad N_{HA}(a, b) = \frac{1}{2}A(a, b) \left[\cos(q) + \frac{q}{\sin(q)} \right],$$

$$(1.6) \quad N_{CA}(a, b) = \frac{1}{2}A(a, b) \left[\cosh(r) + \frac{r}{\sinh(r)} \right],$$

$$(1.7) \quad N_{AC}(a, b) = \frac{1}{2}A(a, b) \left[1 + \frac{2s}{\sin(2s)} \right]$$

and

$$(1.8) \quad \begin{aligned} H(a, b) &< N_{AH}(a, b) < N_{HA}(a, b) < A(a, b) \\ &< N_{CA}(a, b) < N_{AC}(a, b) < C(a, b). \end{aligned}$$

Theorem 1.2. *The double inequalities*

$$(1.9) \quad \alpha_1 A(a, b) + (1 - \alpha_1)H(a, b) < N_{AH}(a, b) < \beta_1 A(a, b) + (1 - \beta_1)H(a, b),$$

$$(1.10) \quad \alpha_2 A(a, b) + (1 - \alpha_2)H(a, b) < N_{HA}(a, b) < \beta_2 A(a, b) + (1 - \beta_2)H(a, b),$$

$$(1.11) \quad \alpha_3 C(a, b) + (1 - \alpha_3)A(a, b) < N_{CA}(a, b) < \beta_3 C(a, b) + (1 - \beta_3)A(a, b),$$

$$(1.12) \quad \alpha_4 C(a, b) + (1 - \alpha_4)A(a, b) < N_{AC}(a, b) < \beta_4 C(a, b) + (1 - \beta_4)A(a, b)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$, $\beta_1 \geq 1/2$, $\alpha_2 \leq 2/3$, $\beta_2 \geq \pi/4 = 0.7853\dots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq \sqrt{3} \log(2 + \sqrt{3})/6 = 0.3801\dots$, $\alpha_4 \leq 2/3$ and $\beta_4 \geq (4\sqrt{3}\pi - 9)/18 = 0.7901\dots$

Theorem 1.3. *The double inequalities*

$$(1.13) \quad \frac{\alpha_5}{H(a, b)} + \frac{1 - \alpha_5}{A(a, b)} < \frac{1}{N_{AH}(a, b)} < \frac{\beta_5}{H(a, b)} + \frac{1 - \beta_5}{A(a, b)},$$

$$(1.14) \quad \frac{\alpha_6}{H(a, b)} + \frac{1 - \alpha_6}{A(a, b)} < \frac{1}{N_{HA}(a, b)} < \frac{\beta_6}{H(a, b)} + \frac{1 - \beta_6}{A(a, b)},$$

$$(1.15) \quad \frac{\alpha_7}{A(a, b)} + \frac{1 - \alpha_7}{C(a, b)} < \frac{1}{N_{CA}(a, b)} < \frac{\beta_7}{A(a, b)} + \frac{1 - \beta_7}{C(a, b)},$$

$$(1.16) \quad \frac{\alpha_8}{A(a, b)} + \frac{1 - \alpha_8}{C(a, b)} < \frac{1}{N_{AC}(a, b)} < \frac{\beta_8}{A(a, b)} + \frac{1 - \beta_8}{C(a, b)},$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_5 \leq 0$, $\beta_5 \geq 2/3$, $\alpha_6 \leq 0$, $\beta_6 \geq 1/3$, $\alpha_7 \leq [2\sqrt{3} - \log(2 + \sqrt{3})]/[2\sqrt{3} + \log(2 + \sqrt{3})] = 0.4490\dots$, $\beta_7 \geq 2/3$, $\alpha_8 \leq (9\sqrt{3} - 4\pi)/(3\sqrt{3} + 4\pi) = 0.1701\dots$ and $\beta_8 \geq 1/3$.

2. LEMMAS

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1. (See [15, Theorem 1.25]) For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on (a, b) , let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) , then so are

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. (See [16, Lemma 1.1]) Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r > 0$ and $a_n, b_n > 0$ for all $n \geq 0$. If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for all $n \geq 0$, then the function $f(x)/g(x)$ is also (strictly) increasing (decreasing) on $(0, r)$.

Lemma 2.3. (See [12, Theorem 4.1]) If $a > b$, then

$$N(b, a) > N(a, b).$$

Lemma 2.4. The function

$$\varphi_1(t) = \frac{\sinh(2t) - 4 \sinh(t) + 2t}{\sinh(2t) - 2 \sinh(t)}$$

is strictly increasing from $(0, \infty)$ onto $(2/3, 1)$.

Proof. Making use of power series expansion we get

$$(2.1) \quad \varphi_1(t) = \frac{\sum_{n=1}^{\infty} \frac{2^{2n+1}-4}{(2n+1)!} t^{2n+1}}{\sum_{n=1}^{\infty} \frac{2^{2n+1}-2}{(2n+1)!} t^{2n+1}} = \frac{\sum_{n=0}^{\infty} \frac{2^{2n+3}-4}{(2n+3)!} t^{2n}}{\sum_{n=0}^{\infty} \frac{2^{2n+3}-2}{(2n+3)!} t^{2n}}.$$

Let

$$(2.2) \quad a_n = \frac{2^{2n+3} - 4}{(2n+3)!}, \quad b_n = \frac{2^{2n+3} - 2}{(2n+3)!}.$$

Then

$$(2.3) \quad a_n > 0, \quad b_n > 0$$

and $a_n/b_n = 1 - 1/(2^{2n+2} - 1)$ is strictly increasing for all $n \geq 0$.

Note that

$$(2.4) \quad \varphi_1(0^+) = \frac{a_0}{b_0} = \frac{2}{3}, \quad \varphi_1(\infty) = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1.$$

Therefore, Lemma 2.4 follows easily from Lemma 2.2 and (2.1)-(2.4) together with the monotonicity of the sequence $\{a_n/b_n\}$. \square

Lemma 2.5. *The function*

$$\varphi_2(t) = \frac{2t - \sin(2t)}{\sin t(1 - \cos t)}$$

is strictly increasing from $(0, \pi/2)$ onto $(8/3, \pi)$.

Proof. Let $f_1(t) = 2t - \sin(2t)$ and $g_1(t) = \sin t(1 - \cos t)$. Then simple computations lead to

$$(2.5) \quad \varphi_2(t) = \frac{f_1(t) - f_1(0)}{g_1(t) - g_1(0)}$$

and $f_1'(t)/g_1'(t) = 4[1 - 1/(2 + 1/\cos t)]$ is strictly increasing on $(0, \pi/2)$.

Note that

$$(2.6) \quad \varphi_2(0^+) = \lim_{t \rightarrow 0^+} \frac{f_1'(t)}{g_1'(t)} = \frac{8}{3}, \quad \varphi_2(\pi/2) = \pi.$$

Therefore, Lemma 2.5 follows from Lemma 2.1, (2.5), (2.6) and the monotonicity of $f_1'(t)/g_1'(t)$. \square

Lemma 2.6. *The function*

$$\varphi_3(t) = \frac{\sinh(t) \cosh(t) - t}{[\sinh(t) \cosh(t) + t](\cosh(t) - 1)}$$

is strictly decreasing from $(0, \infty)$ onto $(0, 2/3)$.

Proof. Simple computations lead to

$$(2.7) \quad \begin{aligned} \varphi_3(t) &= \frac{2 \sinh(2t) - 4t}{\sinh(3t) + 4t \cosh(t) + \sinh(t) - 2 \sinh(2t) - 4t} \\ &= \frac{\sum_{n=0}^{\infty} \frac{2^{2n+4}}{(2n+3)!} t^{2n}}{\sum_{n=0}^{\infty} \frac{3^{2n+3} - 2^{2n+4} + 8n + 13}{(2n+3)!} t^{2n}}. \end{aligned}$$

Let

$$(2.8) \quad a_n = \frac{2^{2n+4}}{(2n+3)!}, \quad b_n = \frac{3^{2n+3} - 2^{2n+4} + 8n + 13}{(2n+3)!}.$$

Then

$$(2.9) \quad a_n > 0, \quad b_n > 0$$

and

$$(2.10) \quad \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{2^{2n+4} (5 \times 3^{2n+3} - 24n - 31)}{(3^{2n+5} - 2^{2n+6} + 8n + 21)(3^{2n+3} - 2^{2n+4} + 8n + 13)} < 0$$

for all $n \geq 0$.

Note that

$$(2.11) \quad \varphi_3(0^+) = \frac{a_0}{b_0} = \frac{2}{3}, \quad \varphi_3(\infty) = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0.$$

Therefore, Lemma 2.6 follows easily from (2.7)-(2.11) and Lemma 2.2. \square

Lemma 2.7. *The function*

$$f(t) = 9 \cos t + \frac{t}{\sin t}$$

is strictly decreasing on the interval $(0, \pi/2)$.

Proof. Let $f_2(t) = 9 \sin t \cos t + t$ and $g_2(t) = \sin t$. Then simple computations lead to

$$(2.12) \quad \begin{aligned} f(t) &= \frac{f_2(t) - f_2(0)}{g_2(t) - g_2(0)}, \\ \frac{f'_2(t)}{g'_2(t)} &= \frac{18 \cos^2 t - 8}{\cos t} \end{aligned}$$

and

$$(2.13) \quad \left[\frac{f'_2(t)}{g'_2(t)} \right]' = -\frac{2 \sin t (9 \cos^2 t + 4)}{\cos^2(t)} < 0$$

for $t \in (0, \pi/2)$.

Therefore, Lemma 2.7 follows easily from (2.12) and (2.13) together with Lemma 2.1. \square

Lemma 2.8. *The function*

$$\varphi_4(t) = \frac{\sin t \cos t - t}{(t + \sin t \cos t)(1 - \cos t)}$$

is strictly decreasing from $(0, \pi/2)$ onto $(-1, -2/3)$.

Proof. Let $f_3(t) = \sin t \cos t - t$ and $g_3(t) = (t + \sin t \cos t)(1 - \cos t)$. Then simple computations lead to

$$(2.14) \quad \varphi_4(t) = \frac{f_3(t)}{g_3(t)} = \frac{f_3(t) - f_3(0)}{g_3(t) - g_3(0)},$$

$$(2.15) \quad \frac{f'_3(t)}{g'_3(t)} = \frac{f'_3(t) - f'_3(0)}{g'_3(t) - g'_3(0)}$$

and

$$(2.16) \quad \frac{f''_3(t)}{g''_3(t)} = \frac{4}{4 - (9 \cos t + \frac{t}{\sin t})}.$$

Note that

$$(2.17) \quad \varphi_4(0^+) = \lim_{t \rightarrow 0^+} \frac{f''_3(t)}{g''_3(t)} = -\frac{2}{3}, \quad \varphi_4\left(\frac{\pi}{2}\right) = -1.$$

Therefore, Lemma 2.8 follows from Lemma 2.1 and Lemma 2.7 together with (2.14)-(2.17). \square

3. PROOFS OF THEOREMS 1.1-1.3

Proof of Theorem 1.1. It follows from (1.1)-(1.3) we clearly see that

$$\begin{aligned} N_{AH}(a, b) &= \frac{1}{2} \left[A(a, b) + \frac{H^2(a, b)}{S_{AH}(a, b)} \right] = \frac{1}{2} A(a, b) \left[1 + (1 - v^2)^2 \frac{p}{\tanh(p)} \right] \\ &= \frac{1}{2} A(a, b) \left[1 + \frac{p}{\tanh(p) \cosh^2(p)} \right] = \frac{1}{2} A(a, b) \left[1 + \frac{2p}{\sinh(2p)} \right], \\ N_{HA}(a, b) &= \frac{1}{2} \left[H(a, b) + \frac{A^2(a, b)}{S_{HA}(a, b)} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}A(a, b) \left[(1 - v^2) + \frac{q}{\sin q} \right] = \frac{1}{2}A(a, b) \left[\cos q + \frac{q}{\sin q} \right], \\
N_{CA}(a, b) &= \frac{1}{2} \left[C(a, b) + \frac{A^2(a, b)}{S_{CA}(a, b)} \right] \\
&= \frac{1}{2}A(a, b) \left[(1 + v^2) + \frac{r}{\sinh(r)} \right] = \frac{1}{2}A(a, b) \left[\cosh(r) + \frac{r}{\sinh(r)} \right], \\
N_{AC}(a, b) &= \frac{1}{2} \left[A(a, b) + \frac{C^2(a, b)}{S_{AC}(a, b)} \right] = \frac{1}{2}A(a, b) \left[1 + (1 + v^2)^2 \frac{s}{\tan(s)} \right] \\
&= \frac{1}{2}A(a, b) \left[1 + \frac{s}{\tan(s) \cos^2 s} \right] = \frac{1}{2}A(a, b) \left[1 + \frac{2s}{\sin(2s)} \right].
\end{aligned}$$

Inequalities (1.8) follows easily from $H(a, b) < A(a, b) < C(a, b)$ and Lemma 2.3 together with the fact that $N_{KL}(a, b)$ is a mean of $K(a, b)$ and $L(a, b)$ for $K(a, b), L(a, b) \in \{H(a, b), A(a, b), C(a, b)\}$. \square

Proof of Theorem 1.2. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b) \in (0, 1)$, $p \in (0, \infty)$, $q \in (0, \pi/2)$, $r \in (0, \log(2 + \sqrt{3}))$ and $s \in (0, \pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$. Then from (1.4)- (1.7) we have

$$\begin{aligned}
(3.1) \quad \frac{N_{AH}(a, b) - H(a, b)}{A(a, b) - H(a, b)} &= \frac{\frac{1}{2} \left[1 + \frac{2p}{\sinh(2p)} \right] - (1 - v^2)}{v^2} \\
&= \frac{\frac{1}{2} \left[1 + \frac{2p}{\sinh(2p)} \right] - \frac{1}{\cosh(p)}}{1 - \frac{1}{\cosh(p)}} = \varphi_1(p),
\end{aligned}$$

$$\begin{aligned}
(3.2) \quad \frac{N_{HA}(a, b) - H(a, b)}{A(a, b) - H(a, b)} &= \frac{\frac{1}{2} \left[\cos q + \frac{q}{\sin q} \right] - (1 - v^2)}{v^2} \\
&= \frac{\frac{1}{2} \left[\cos q + \frac{q}{\sin q} \right] - \cos q}{1 - \cos q} = \frac{1}{4} \varphi_2(q),
\end{aligned}$$

$$\begin{aligned}
(3.3) \quad \frac{N_{CA}(a, b) - A(a, b)}{C(a, b) - A(a, b)} &= \frac{\frac{1}{2} \left[\cosh(r) + \frac{r}{\sinh(r)} \right] - 1}{v^2} \\
&= \frac{\frac{1}{2} \left[\cosh(r) + \frac{r}{\sinh(r)} \right] - 1}{\cosh(r) - 1} = \frac{1}{2} \varphi_1(r).
\end{aligned}$$

$$\begin{aligned}
(3.4) \quad \frac{N_{AC}(a, b) - A(a, b)}{C(a, b) - A(a, b)} &= \frac{\frac{1}{2} \left[1 + \frac{2s}{\sin(2s)} \right] - 1}{v^2} \\
&= \frac{\frac{1}{2} \left[1 + \frac{2s}{\sin(2s)} \right] - 1}{\sec(s) - 1} = \frac{1}{4} \varphi_2(s),
\end{aligned}$$

where the functions φ_1 and φ_2 are defined as in Lemmas 2.4 and 2.5, respectively.

Note that

$$(3.5) \quad \varphi_1[\log(2 + \sqrt{3})] = \sqrt{3} \log(2 + \sqrt{3})/6$$

and

$$(3.6) \quad \varphi_2\left(\frac{\pi}{3}\right) = \frac{8\sqrt{3}\pi - 18}{9}.$$

Therefore, inequality (1.9) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$ and $\beta_1 \geq 1/2$ follows from (3.1) and Lemma 2.4, inequality (1.10) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 2/3$ and $\beta_2 \geq \pi/4$ follows from (3.2) and Lemma 2.5, inequality (1.11) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/3$ and $\beta_3 \geq \sqrt{3}\log(2 + \sqrt{3})/6$ follows from (3.3) and (3.5) together with Lemma 2.4, and inequality (1.12) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 2/3$ and $\beta_4 \geq (4\sqrt{3}\pi - 9)/18$ follows from (3.4) and (3.6) together with Lemma 2.5. \square

Proof of Theorem 1.3. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b) \in (0, 1)$, $p \in (0, \infty)$, $q \in (0, \pi/2)$, $r \in (0, \log(2 + \sqrt{3}))$ and $s \in (0, \pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$. Then from (1.4)-(1.7) we have

$$(3.7) \quad \frac{\frac{1}{N_{AH}(a,b)} - \frac{1}{A(a,b)}}{\frac{1}{H(a,b)} - \frac{1}{A(a,b)}} = \frac{\frac{2}{1+\frac{2p}{\sinh(2p)}} - 1}{\frac{1}{1-v^2} - 1} = \frac{\frac{2\sinh(2p)}{2p+\sinh(2p)} - 1}{\cosh(p) - 1} = \varphi_3(p),$$

$$(3.8) \quad \frac{\frac{1}{N_{HA}(a,b)} - \frac{1}{A(a,b)}}{\frac{1}{H(a,b)} - \frac{1}{A(a,b)}} = \frac{\frac{2}{\cos(q)+\frac{q}{\sin(q)}} - 1}{\frac{1}{1-v^2} - 1} = \frac{\frac{2\sin(q)-\sin(q)\cos(q)-q}{\sin(q)\cos(q)+q}}{\frac{1-\cos(q)}{\cos(q)}} = 1 + \varphi_4(q),$$

$$(3.9) \quad \frac{\frac{1}{N_{CA}(a,b)} - \frac{1}{C(a,b)}}{\frac{1}{A(a,b)} - \frac{1}{C(a,b)}} = \frac{\frac{2}{\cosh(r)+\frac{r}{\sinh(r)}} - \frac{1}{1+v^2}}{1 - \frac{1}{1+v^2}} = \varphi_3(r)$$

and

$$(3.10) \quad \frac{\frac{1}{N_{AC}(a,b)} - \frac{1}{C(a,b)}}{\frac{1}{A(a,b)} - \frac{1}{C(a,b)}} = 1 + \varphi_4(s),$$

where the functions φ_3 and φ_4 are defined as in Lemmas 2.6 and 2.8, respectively.

Note that

$$(3.11) \quad \varphi_3[\log(2 + \sqrt{3})] = \frac{2\sqrt{3} - \log(2 + \sqrt{3})}{2\sqrt{3} + \log(2 + \sqrt{3})}$$

and

$$(3.12) \quad \varphi_4\left(\frac{\pi}{3}\right) = -\frac{8\pi - 6\sqrt{3}}{4\pi + 3\sqrt{3}}.$$

Therefore, Theorem 1.3 follows easily from (3.7)-(3.12) together with Lemmas 2.6 and 2.8. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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